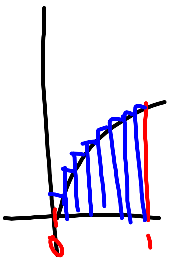
# Definite Integrals

Warm-up:

Consider the region bounded by the graph of $f(x) = \sqrt{x}$ and the $x$-axis for $0 \le x \le 1$. Evaluate the limit $\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x_i$ where $c_i$ is the right endpoint of the partition given by $c_i = \frac{i^2}{n^2}$ and $\Delta x_i$ is the width of the $i$th interval.



$$\Delta x = \frac{i^2}{n^2} - \frac{(i-1)^2}{n^2} = \frac{i^2 - (i^2 - 2i + 1)}{n^2} = \frac{2i-1}{n^2}$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{\frac{i^2}{n^2}} \left(\frac{2i-1}{n^2}\right) = \frac{i}{n}\left(\frac{2i-1}{n^2}\right) = \frac{2i^2 - i}{n^3}$$

$$= \lim_{n\to\infty} \frac{1}{n^3}\left[\sum_{i=1}^{n} 2i^2 - \sum_{i=1}^{n} i\right]$$

$$= \lim_{n\to\infty} \frac{1}{n^3}\left[\frac{2n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2}\right]$$

$$= \lim_{n\to\infty} \frac{1}{n^3}\left[\frac{2n(2n^2+3n+1)}{6} - \frac{n^2+n}{2}\right]$$

$$= \left[\frac{4n^3+6n^2+2n}{6} - \frac{3}{3}\cdot\frac{n^2+n}{2}\right]$$

$$= \frac{4n^3+6n^2+2n-3n^2-3n}{6n^3}$$

$$= \frac{4n^3+3n^2-n}{6n^3} = \lim_{n\to\infty}\left[\frac{4n^3}{6n^3} + \frac{3n^2}{6n^3} - \frac{n}{6n^3}\right]$$

$$= \lim_{n\to\infty}\left[\frac{2}{3} + \cancel{\frac{1}{2n}} - \cancel{\frac{1}{6n^2}}\right]$$

$$= \frac{2}{3}$$

## Definition of a Definite Integral

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x_i = \int_a^b f(x)dx$$

If a function $f$ is continuous on the closed interval $[a, b]$, then $f$ is integrable on $[a, b]$

## Definite Integral as the Area of a Region

$$\text{Area} = \int_a^b f(x)dx$$

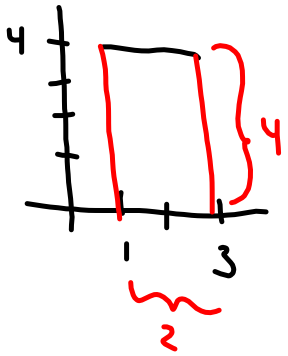
## Areas of Common Geometric Figures

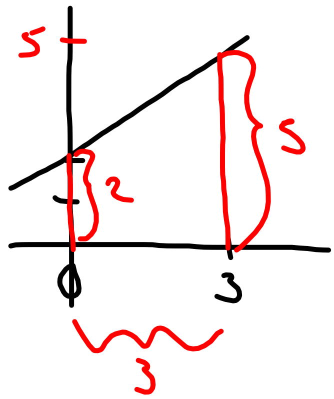
Evaluate each integral using a geometric formula

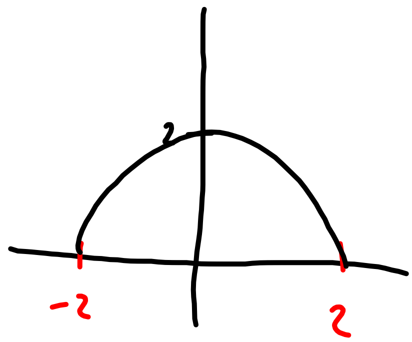
a. $\int_1^3 4\,dx = 8$

b. $\int_0^3 (x+2)dx$

c. $\int_{-2}^{2} \sqrt{4-x^2}\,dx$



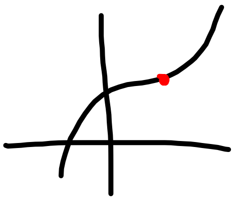

$$A = \frac{\pi r^2}{2} = \frac{\pi(2)^2}{2} = 2\pi$$

$$A = \frac{1}{2}(b_1+b_2)h$$

$$= \frac{(2+5)3}{2} = \frac{21}{2}$$

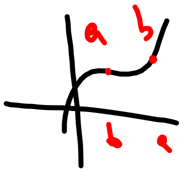
## Properties of Definite Integrals

If $f$ is defined at $x = a$, then



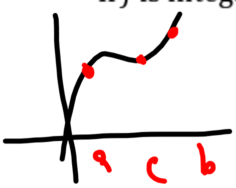
$$\int_a^a f(x)dx = 0$$

If $f$ is integrable on $[a, b]$, then




$$\int_b^a f(x)dx = -\int_a^b f(x)dx$$

If $f$ is integable on the three closed intervals determined by $a$, $b$, and $c$, then




$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

$$\int_a^b kf(x)dx = k\int_a^b f(x)dx$$

$$\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$$

## Evaluating a Definite Integral

Evaluate $\int_1^3 (-x^2 + 4x - 3)dx$ using each of the following values

$$\int_1^3 x^2 dx = \frac{26}{3} \qquad \int_1^3 x dx = 4 \qquad \int_1^3 dx = 2$$

$$\int_1^3 (-x^2+4x-3)dx = \int_1^3 -x^2 dx + \int_1^3 4x dx - \int_1^3 3 dx$$

$$= -\int_1^3 x^2 dx + 4\int_1^3 x dx - 3\int_1^3 dx$$

$$= -\frac{26}{3} + 4(4) - 3(2)$$

$$= -\frac{26}{3} + 16 - 6 = -\frac{26}{3} + 10 = \frac{-26}{3} + \frac{30}{3} = \frac{4}{3}$$