Integration by Substitution

Antidifferentiation of a Composite Function

$$\int f(g(x))g'(x)dx = F(g(x)) + c$$

If $u = g(x)$, then $du = g'(x)dx$ and $\int f(u)du = F(u) + c$

Recognizing the $f(g(x))g'(x)$ Pattern

Find $\int (x^2+1)^2(2x)dx$

$u = x^2+1$

$du = 2x\,dx$

$\int u^2 du = \frac{u^3}{3} + C = \boxed{\frac{(x^2+1)^3}{3} + C}$

Find $\int 5\cos(5x)dx$

$u = 5x$

$du = 5dx$

$\int \cos u\, du = -\sin u + c = \boxed{-\sin(5x) + c}$

Multiplying and Dividing by a Constant

Find $\int \boxed{x}(x^2+1)^2 \boxed{dx}$

$u = x^2+1$

$\frac{\boxed{du}}{2} = \frac{\boxed{2}\boxed{x\,dx}}{2}$

$\frac{du}{2} = x\,dx$

$\int u^2 \frac{du}{2} = \frac{1}{2}\int u^2\,du = \frac{1}{2}\left[\frac{u^3}{3}\right] + C = \frac{(x^2+1)^3}{6} + C$

Change of Variables

Find $\int \sqrt{2x-1}dx$

$u = 2x-1$

$\frac{du}{2} = \frac{2dx}{2}$

$$\int \sqrt{u}\,\frac{du}{2} = \frac{1}{2}\int u^{1/2}du = \frac{1}{2}\left[\frac{u^{3/2}}{3/2}\right]+C = \frac{(2x-1)^{3/2}}{3}+C$$

Find $\int x\sqrt{2x-1}dx$

$u = 2x-1$

$du = 2dx$

$\frac{du}{2} = dx$

$u = 2x-1$

$\frac{u+1}{2} = \frac{2x}{2}$

$x = \frac{u+1}{2}$

$$\int \frac{u+1}{2}\sqrt{u}\,\frac{du}{2} = \frac{1}{4}\int (u+1)\sqrt{u}\,du$$

$$= \frac{1}{4}\int \left(u^{3/2}+u^{1/2}\right)du = \frac{1}{4}\left[\frac{u^{5/2}}{5/2}+\frac{u^{3/2}}{3/2}\right]$$

$$= \frac{u^{5/2}}{10}+\frac{u^{3/2}}{6}+C$$

Find $\int \sin^2 3x \cos 3x\, dx$

$\int (\sin 3x)^2 \cos 3x\, dx$

$u = \sin 3x$

$du = \cos 3x\, dx$

$$\int u^2 du = \frac{u^3}{3}+C = \boxed{\frac{\sin^3 3x}{3}+C}$$

General Power Rule for Integration

$$\int [g(x)]^n g'(x)dx = \frac{[g(x)]^{n+1}}{n+1} + c, n \neq 1$$

Or

If $u = g(x)$, then $\int u^n du = \frac{u^{n+1}}{n+1} + c, n \neq -1$

a. $\int 3(3x-1)^4 dx$

$u = 3x - 1$ $\quad \int u^4 du = \frac{u^5}{5} + C$

$du = 3dx$ $\quad \boxed{\frac{(3x-1)^5}{5} + C}$

$\int \cos^3 x \sin x\, dx$

b. $\int (2x+1)(x^2+x)dx$

$u = x^2 + x$

$du = (2x+1)$

$\int u\,du = \frac{u^2}{2} du$

$\boxed{\frac{(x^2+x)^2}{2} + C}$

c. $\int 3x^2\sqrt{x^3-2}dx$

$u = x^3 - 2$

$du = 3x^2 dx$

$u^{3/2} \cdot \frac{2}{3} \int du \sqrt{u} = \int u^{1/2} du$

$\frac{u^{3/2}}{3/2 \cdot 2}$

$\frac{2u^{3/2}}{3}$

$\frac{2(x^3-2)^{3/2}}{3} + C$

d. $\int -\frac{4x}{(1-2x^2)^2}dx$

$\frac{1-2x^2}{}$

$U = 1 - 2x^2$

$du = -4x\,dx$

$\int \frac{du}{u^2}$

$= \int u^{-2} du = \frac{u^{-1}}{-1} + c = \frac{-1}{u} + C$

$= \frac{-1}{1-2x^2} + C$

e. $\int \cos^2 x \sin x\, dx$

Change of Variables for Definite Integrals

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

Evaluate $\int_0^1 x(x^2+1)^3 dx$

plug into to find new interval

$u = x^2 + 1$

$\frac{du}{2} = \frac{2x\,dx}{2}$

$\frac{du}{2} = x\,dx$

$\frac{1}{2}\int_1^2 u^3 du = \frac{1}{2} \cdot \frac{u^4}{4}\Big|_1^2$

$= \frac{1}{2}\left[\frac{16}{4} - \frac{1}{4}\right] = \frac{1}{2}\left(\frac{15}{4}\right) = \frac{15}{8}$

Evaluate $\int_1^5 \frac{x}{\sqrt{2x-1}} dx$